COMPARISON OF THE ROBUSTNESS OF LQ CONTROLLERS IN THE POSITION AND INCREMENTAL FORM

Vladimír Burlak, Jakub Dokoupil

Doctoral Degree Programme (1, 2), FEEC BUT

E-mail: xburla00@stud.feec.vutbr.cz, xdokou05@stud.feec.vutbr.cz

Supervised by: Petr Pivoňka

E-mail: pivonka@feec.vutbr.cz

Abstract: Linear Quadratic optimal controller (LQ) can be used in the positional and incremental form. This article deals with the comparison of the robustness against parameters deviations of the regulated system with firmly set up LQ controllers in the positional and incremental form. The sensitivity analysis was used for this comparison. The article also contains the deduction of effective calculations of the sensitivity function and the simulations results.

Keywords: LQ, optimal control, sensitivity analysis, sensitivity function

1. INTRODUCTION

Robustness against parameters changes of the system is important for the regulation with fixed controller. The same demand can be applied for the adaptive control with on-line identification of this regulated system. If there appears a quick change of dynamic parameters then it is necessary to overcome the time during which the on-line identification does not provide sufficiently exact model. One of the possible solutions can be to let the controller set up on the same parameters and wait for sufficiently exact system model. Adaptive and fixed controllers are equivalent during this time.

Linear Quadratic optimal controller (LQ) can be used in the positional and incremental form. This article deals with the comparison of the robustness against parameters deviations of the regulated system with firmly set up LQ controllers in the positional and incremental form. The sensitivity analysis was used for this comparison. The article also contains the deduction of effective calculations of the sensitivity function and the simulations results.

2. STATE SPACE REPREZENTATION AND CONTROL LAW

2.1. PSEUDO STATE SPACE REPREZENTATION

Pseudo state space representation is used for discrete dynamic systems. State space vector is composed of delayed values of input and output control system. State space equations can be found for example in [3]. State space representation uses the parameters of transfer function. This vector can have the form (2.2). Transfer function of the same system is in the form (2.1).

$$F_D(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \tag{2.1}$$

$$\theta^T = [b_1 \quad \dots \quad b_m \quad -a_1 \quad \dots \quad -a_n] = [\theta_1 \quad \dots \quad \theta_{m+n}] \tag{2.2}$$

This reprezentation has all state variables measurable. This is great advantage.

2.2. STATE SPACE REPRESENTATION OF SYSTEM

State equations have for position controller the form (2.3)

$$\begin{aligned} x(k+1,\theta) &= A(\theta)x(k,\theta) + B(\theta)u(k) \\ y(k,\theta) &= C(\theta)x(k,\theta) + D(\theta)u(k) \end{aligned} \tag{2.3}$$

where $x(k,\theta)$ is the vector of state variables in step k, $u(k)$ is system input, $y(k,\theta)$ is system output and θ is parameter vector in the form (2.2). State equations have for incremental controller the form (2.4)

$$\begin{aligned} \begin{bmatrix} x(k+1,\theta) \\ u(k+1) \end{bmatrix} &= \begin{bmatrix} A(\theta) & B(\theta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k,\theta) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u(k) \\ y(k,\theta) &= [C(\theta) \quad D(\theta)] \begin{bmatrix} x(k,\theta) \\ u(k) \end{bmatrix} \end{aligned} \tag{2.4}$$

where

$$\Delta u(k) = u(k+1) - u(k). \tag{2.5}$$

If $D(\theta)$ is the zero matrix, then we can use the equation (2.6) for both forms.

$$\begin{aligned} x_q(k+1,\theta) &= A_q(\theta)x_q(k,\theta) + B_q u_q(k) \\ y(k,\theta) &= C_q(\theta)x_q(k,\theta) \end{aligned} \tag{2.6}$$

2.3. LINEAR QUADRATIC OPTIMAL CONTROLLER

Discrete Linear Quadratic (LQ) optimal controller is state controller. Parameters of the discrete LQ controller are determined by minimizing the quadratic criterion for linear time-invariant systems. This criterion is in the form (2.7)

$$J = \sum_{i=0}^{\infty} x_q^T(i)Qx_q(i) + u_q^T(i)Ru_q(i) \tag{2.7}$$

where positive semidefinite matrix Q scales deviations of state variables $x_q(i)$ and positive definite matrix R scales control energy. The feedback matrix K is the result of minimizing the value of the criterial function (2.7). The matrix K is solving by equation (2.8).

$$K = \left(R + B_q^T P B_q\right)^{-1} B_q^T P A_q \tag{2.8}$$

where

$$P = Q + K^T R K + \left(A_q - B_q K\right)^T P \left(A_q - B_q K\right). \tag{2.9}$$

The control law is given by (2.10).

$$u_q(k) = -Kx_q(k) \tag{2.10}$$

If the matrix is without depending on the vector θ, then the matrix contains the parameters $\hat{\theta}$. These parameters are used for design of controller parameters. LQ control and solution of the equation (2.9) can be found in [1,2].

3. SENSITIVITY FUNCTION

Sensitivity analysis was used to verify the robustness of LQ controllers in the position and the incremental form. It was necessary to deduce sensitivity function. This function defines the relationship between the change of parameters θ and the value of criterial function $J(\theta)$ in the form (2.7). The derivation assumes the following: the parameters are constant during control (constant deviations), controllers are fixed. Approximation of the sensitivity function can be defined by (3.1)

$$\Delta J(\theta) \approx \sum_{j=1}^{m+n} \frac{\partial J(\theta)}{\partial \theta_j} \Delta\theta_j \tag{3.1}$$

where $\Delta J(\theta)$ is deviation of the value of criterial function and $\Delta\theta_j$ is deviation of the j-th parameter. If we use relations (2.10) and (2.6), then the first equation has the form (3.2).

$$x_q(k+1,\theta) = \big(A_q(\theta) - B_q K\big)x_q(k,\theta) \tag{3.2}$$

Partial derivatives of criterial function (2.7) according to the j-th parameter is given by equation (3.3).

$$\begin{aligned}\frac{\partial J(\theta)}{\partial \theta_j} &= \sum_{i=0}^{\infty} \frac{\partial \left(x_q^T(i)(Q + K^T R K)x_q(i)\right)}{\partial \theta_j} = \\ &= 2\sum_{i=0}^{\infty} \frac{\partial x_q^T(i)}{\partial \theta_j}(Q + K^T R K)x_q(i)\end{aligned} \tag{3.3}$$

Partial derivative of $x_q(k+1,\theta)$ according to the j-th parameter is given by equation (3.4).

$$\frac{\partial x_q(k+1,\theta)}{\partial \theta_j} = \frac{\partial A_q(\theta)}{\partial \theta_j} x_q(k,\theta) + \big(A_q(\theta) - B_q K\big)\frac{\partial x_q(k,\theta)}{\partial \theta_j} \tag{3.4}$$

If we use equation (2.6) and (2.10), then we obtain an autonomous dynamical system with the first state equation in the form (3.5)

$$\begin{aligned}&\begin{bmatrix} x_q(k,\theta) \\ \dfrac{\partial x_q(k,\theta)}{\partial \theta_j} \end{bmatrix} = \begin{bmatrix} A_q(\theta) - B_q K & 0 \\ \dfrac{\partial A(\theta)}{\partial \theta_j} & A_q(\theta) - B_q K \end{bmatrix}^k \begin{bmatrix} x_q(0) \\ \dfrac{\partial x_q(0)}{\partial \theta_j} \end{bmatrix} = \\ &= E^k(\theta)x_l(0) = x_l(k,\theta)\end{aligned} \tag{3.5}$$

with initial condition (3.6).

$$x_l(0) = \begin{bmatrix} x_q(0) \\ 0 \end{bmatrix} \tag{3.6}$$

Furthermore, we introduce the substitution (3.7)

$$S = \begin{bmatrix} 0 \\ I \end{bmatrix}(Q + K^T R K)[I \quad 0] \tag{3.7}$$

where I is the identity matrix and 0 is zero matrix. Both matrices are square. Their dimensions are equal as dimension matrix $A_q(\theta)$. Now we can rewrite equation (3.3) to form (3.8).

$$\frac{\partial J(\theta)}{\partial \theta_j} = 2\sum_{i=0}^{\infty} \frac{\partial x_q^T(i)}{\partial \theta_j}(Q + K^T RK)x_q(i) =$$
$$= 2x_l^T(0)\sum_{i=0}^{\infty}\left[\left(E^i(\theta)\right)^T SE^i(\theta)\right]x_l(0) \tag{3.8}$$

or

$$\frac{\partial J(\theta)}{\partial \theta_j} = 2x_l^T(0)P_S(\theta)x_l(0) \tag{3.9}$$

where

$$P_S(\theta) = S + E^T(\theta)P_S(\theta)E(\theta). \tag{3.10}$$

Equation (3.10) has the same form as equation (2.9). Methods of solving equation (3.10) can be found in [1,2,3].

4. SIMULATION RESULTS

Transfer function of the system was assumed in the form (4.1).

$$F(s) = \frac{K_s}{T^2 s^2 + 2T\gamma s + 1} \tag{4.1}$$

Every simulation consisted of the following:

The parameters T, γ and gain K_S were set to $T = 100s$, $\gamma = 1$ and $K_S = 10$. The system was discretized. Controller parameters were set to this system and the criterial function was calculated by (2.7). Furthermore, parameters have been changed, the system was discretized and the value of the deviation parameter was calculated. Further deviation criterial function was calculated by (3.1). In conclusion, change in the relative value of criterial function was calculated according to equation (4.2)

$$\delta = 100\frac{|\Delta J(\theta)|}{J(\hat{\theta})} \tag{4.2}$$

where $\hat{\theta}$ is a parameter vector, to which the controller was designed. The sampling period was set to 0.1s. The parameter R was set to one. Matrix Q was zero matrix, except the coefficient $Q_{3,3}$. Its value was set to one. Vector (4.3) was chosen as an initial condition for the LQ controller in positional form. Vector (4.4) was chosen as an initial condition for the LQ controller in incremental form.

$$x_{q\ position}(0) = [0{,}1 \quad 0{,}1 \quad 1 \quad 1\ ]^{\mathrm{T}} \tag{4.3}$$

$$x_{q\ incremantal}(0) = [0{,}1 \quad 0{,}1 \quad 1 \quad 1 \quad 0\ ]^{\mathrm{T}} \tag{4.4}$$

Table 4.1 contains the results. In all cases, the controller in the form of incremental changes reached a lower relative deviation of the criterial function. The results show that the LQ controller in incremental form is more robust in particular changes in the time constant of the system.

#	γ	K_s	T[s]	$\delta_{incremental}$ [%]	$\delta_{position}$ [%]
1	1,0	10,0	100	0,00	0,00
2	1,0	10,0	105	3,66	5,49
3	1,0	10,0	110	8,16	11,99
4	0,5	10,0	105	4,37	7,92
5	0,5	10,0	110	9,41	16,43
6	1,0	10,5	100	1,48	2,29
7	1,0	11,0	100	2,72	4,27
8	0,5	10,5	100	1,29	1,76
9	0,5	11,0	100	2,61	3,98

Table 4.1 Simulation results

5. CONCLUSION

This work dealt with the robustness of firmly set up LQ controllers. The aim of the work was also the carry out the comparison of controller robustness in the positional and incremental form. Relative change of the value of criterial function was used as the measure for this comparison. The sensitivity analysis was used for these comparisons. For its usage there was necessary to deduct the efficient calculation of sensitivity function, the final calculation of sensitivity function can be performed by using the relations (3.1), (3.5), (3.7), (3.9) a (3.10).

When watching these simulated results it is apparent that for changes of both time constants and for strengthening the system the relative value gained the value of criterial function which had afterwards significantly lower value from the increasing module. This phenomenon was mostly apparent for changes of the time constant. LQ controller in the incremental form reached a lower value of relative changes in test functions.

ACKNOWLEDGEMENT

This work was supported by grant „Modern Methods and Approaches in Automation“ from the Internal Grant Agency of Brno University of Technology (grant No. FEKT-S-10-12).

REFERENCES

[1] SINHA, A.: Linear systems, optimal and robust control, New York, CRC Press, ISBN 0-8493-9217-9

[2] PIVOŇKA, P.: Optimalizace regulátorů. VUT Brno, skriptum 2005.

[3] LORENC, V.: *LQ regulátor* (učební text). UAMT VUT Brno, 2007.